Two-epoch optimal design of displacement monitoring networks

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Abstract

In the traditional method of optimal design of displacement monitoring networks a higher precision, $\sqrt{2}$ times better than the desired accuracy of displacements, is considered for the net points in such a way that the accuracy of the detected displacements meets the desired one. However, in this paper, we develop an alternative method by considering the total number of observations in two epochs without such a simple assumption and we call it two-epoch optimisation. This method is developed based on the Gauss-Helmert adjustment model and the variances of the observations are estimated instead of the weights to optimise the observation plan. This method can deliver the same results as the traditional one, but with less required observations in each epoch. In the optimal design of a monitoring network, the accuracies of the displacements and/or deformation parameters are also considered as some criteria. Displacements are important for monitoring networks as the deformation parameters are estimated from them, if the monitored object can be considered as a continuum medium. However, not all objects are of continuum nature, in such a case displacements in different parts of the object are considered instead of deformation parameters. So far, geodetic networks have been designed based on the precision and/or reliability of the network for one epoch of observations. In fact, in displacement monitoring networks the goal is to determine the displacements or the coordinate differences between two epochs of observations. Considering two epochs for optimising a displacement monitoring network is a new issue, which we will discuss in this study based on the precision of the estimated displacements. Here, we will discuss both methods of one-epoch and two-epoch optimisation and compare them in theoretical and practical view.

Optimisation principles

In the traditional way, the network is designed in such a way that the precision of the coordinates of the net points becomes $\sqrt{2}$ times more so that the precision of the displacements reaches to the desired accuracy. In this case the variance-covariance (VC) matrix of the displacement is:

$$\mathbf{C}_{\Delta\hat{\mathbf{x}}} = 2\mathbf{N}^{-1} - \mathbf{HE}^{-1}\mathbf{H} \quad \text{where} \quad \mathbf{N} = \mathbf{A}^{\mathrm{T}}\mathbf{PA} + \mathbf{DD}^{\mathrm{T}} \quad \text{and} \quad \mathbf{E} = \mathbf{H}^{\mathrm{T}}\mathbf{DD}^{\mathrm{T}}\mathbf{H}$$

A is the design matrix, **P** stands for the weight matrix, **D** is the matrix carrying the datum information and **H** is the same matrix which is used for adjustment using inner constraints. The main idea of the one-epoch optimisation is to change the configuration and determine the weight of observations by fitting the following mathematical model, which is in fact the Taylor expansion of the VC matrix, to the desired VC matrix of displacements:

$$\mathbf{C}_{\Delta\hat{\mathbf{x}}} = \mathbf{C}^{0}_{\Delta\hat{\mathbf{x}}} + \sum_{i=1}^{m}\left(\frac{\partial \mathbf{C}_{\Delta\hat{\mathbf{x}}}}{\partial x_i}\Delta x_i + \frac{\partial \mathbf{C}_{\Delta\hat{\mathbf{x}}}}{\partial y_i}\Delta y_i\right) + \sum_{j=1}^{n}\frac{\partial \mathbf{C}_{\Delta\hat{\mathbf{x}}}}{\partial p_j}\Delta p_j \quad \text{where} \quad \frac{\partial \mathbf{C}_{\Delta\hat{\mathbf{x}}}}{\partial p_j} = \mathbf{N}^{-1}\mathbf{A}^{\mathrm{T}}\frac{\partial \mathbf{P}}{\partial p_j}\mathbf{AN}^{-1}$$

where $\mathbf{C}_{\Delta\hat{\mathbf{x}}}$ is the desired VC matrix of displacement vector $\Delta\hat{\mathbf{x}}$, $\mathbf{C}^{0}_{\Delta\hat{\mathbf{x}}}$ the initial VC matrix, Δx_i, Δy_i and Δp_j are the improvements to the coordinates and the observation weights. m and n are the number of net points and observations, respectively. In order to solve this problem quadratic programming should be used (see Eshagh and Alizadeh 2014).

For two-epoch optimisation, we use the Gauss-Helmert model for estimating the displacement. The resulted displacement are the same with those obtained from the Gauss-Markov model, but this model can consider all observations in two epochs rather than one. The VC matrix of the displacements are:

$$\mathbf{C}_{\Delta\hat{\mathbf{x}}} = \mathbf{M}^{-1} - \mathbf{HE}^{-1}\mathbf{H} \quad \text{where} \quad \mathbf{K} = \mathbf{BQB}^{\mathrm{T}},\ \mathbf{Q} = \operatorname{diag}\begin{pmatrix}\mathbf{Q}_1 & \mathbf{Q}_2\end{pmatrix} \text{ and } \mathbf{M} = \mathbf{A}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{A} + \mathbf{DD}^{\mathrm{T}}$$

Furthermore $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the VC matrices of observations in the epochs 1 and 2, respectively. Again, we change the configuration and precision of the observations in such a way that the present VC matrix of the displacement is fitted to the desired one.

$$\mathbf{C}_{\Delta\hat{\mathbf{x}}} = \mathbf{C}^{0}_{\Delta\hat{\mathbf{x}}} + \sum_{i=1}^{m}\left(\frac{\partial \mathbf{C}_{\Delta\hat{\mathbf{x}}}}{\partial x_i}\Delta x_i + \frac{\partial \mathbf{C}_{\Delta\hat{\mathbf{x}}}}{\partial y_i}\Delta y_i\right) + \sum_{j=1}^{2n}\frac{\partial \mathbf{C}_{\Delta\hat{\mathbf{x}}}}{\partial q^k_j}\Delta q^k_j \qquad k = \begin{cases}1 & j \le n\\ 2 & j > n\end{cases}$$

$$\frac{\partial \mathbf{C}_{\Delta\hat{\mathbf{x}}}}{\partial q_j} = \mathbf{M}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{B}\frac{\partial \mathbf{Q}}{\partial q_j}\mathbf{B}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{AM}^{-1}$$

where q^k_j stands for the variance of observations at epoch k. k = 1 means the first epoch and k = 2 is the second one. As we observe j ranges from 1 to 2n, because all observations of two epochs are considered at once.

Numerical test

Figure 1a shows the optimised network by the one-epoch method on the background of the network prior to optimisation. As it is seen, the lengths L26, L32 and L 56 have been deleted from the observation plan, which means that the network is able to deliver displacement with the desired accuracy based on one-epoch method with 3 observations less than the planed one. All error ellipses of the displacements have been presented 5000 times larger than their true values for a better visualisation. They became larger after optimisation, which means that a precision of 3 mm for the displacement is achievable with 3 less observations.

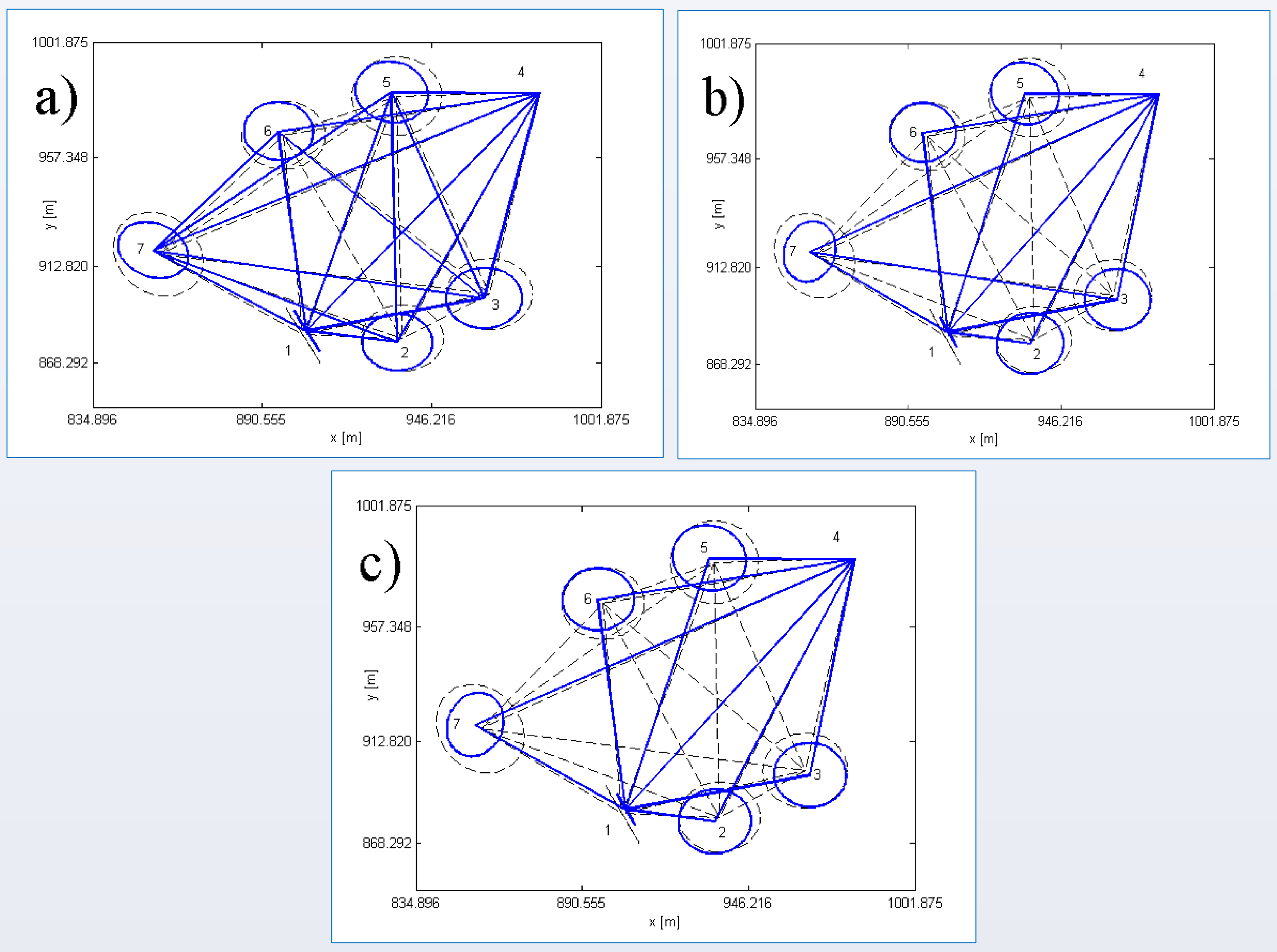

Figure 1. Network configuration and error ellipses of the displacements before and after optimisation. a) Optimised network based on one-epoch optimisation, b) and c) two-epoch optimisation of network in Epochs 1 and 2, respectively.

Figures 1b and 1c show the optimised network based on the two-epoch method. The former presents the optimised one in Epoch 1 and the latter in Epoch 2. In the one-epoch optimisation, we considered a higher precision for the displacements by dividing the desired precession by $\sqrt{2}$ but in the two-epoch optimisation we do not have to do that as we can consider the observation plans of both epochs together and optimise them simultaneously. Lots of observations have been removed from the plan in the two-epoch optimisation. Also, we observe that the plan in Epoch 2 has one observation less than that in Epoch 1. This means that measuring the same quantity as those in Epoch 1 is not necessary. The figure shows that in order to attain the accuracy of 3 mm for the displacement at each point, observing two lengths from the reference points is enough. However, this is not a general conclusion, but valid for the present network. One may say that the observations to the reference points are preserved.

Table 1. Displacement error of net points before and after optimisation.

Point No.	*Before Optimisation*		*One-Epoch Optimisation*		*Two-Epoch Optimisation*	
	$\sigma_{\Delta x}$ (mm)	$\sigma_{\Delta y}$ (mm)	$\sigma_{\Delta x}$ (mm)	$\sigma_{\Delta y}$ (mm)	$\sigma_{\Delta x}$ (mm)	$\sigma_{\Delta y}$ (mm)
1	2	2	1	2	1	1
2	3	3	3	3	3	3
3	3	4	3	3	3	3
4	0	0	0	0	0	0
5	4	3	3	3	3	3
6	3	3	3	3	3	3
7	3	4	2	3	2	3

Table 1 represents the errors of the displacements before and after optimisation. Since the network is designed for displacement monitoring purpose, therefore, the variances of all net points should be 2 times larger for the displacements.

The table presents that the errors of the displacements are more or less about 3 mm before optimisation except for Point 1 because the direction of 4 to 1 is kept fixed in the network. If we assume that this network is optimised in such a way that the errors of displacements become 3 mm based on the one-epoch approach, we have to select the desired error of each point $\sqrt{2}$ times smaller. As the table, shows both one- and two-epoch approaches can deliver the same accuracies for the displacements. However, in the two-epoch approach, 9 observations are removed from the plan of each epoch, which lead to loss of redundancy and accuracy of the coordinates of the points. Nevertheless, the accuracies of the displacements are preserved below the acceptable level. Another issue is that the weight of one observation may come out very large in one epoch with respect to another, in such a case, the use of highest weight is recommended as our goal in the two-epoch optimisation is to design the network and determine which observations at which epoch should be kept or deleted from the plan.

Conclusions

The developed two-epoch optimisation of displacement monitoring network delivers the same accuracies for the displacement as those of the traditional one-epoch method, but less observations are used in this approach. The configuration and position changes of the points are the same in both approaches. In short, one- and two-epoch approaches both delivers similar accuracies of displacements and configuration, but the latter uses less observations in each epoch. Therefore, the two-epoch approach is more economical and practicable than the traditional one. One point that should be stated here is that the weight of one observation may come out considerably larger/smaller in one epoch than another. However, this point will not be significant if we consider the larger weight for that observation in both epochs.

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