Regional gravity field modeling: A comparison of methods

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Gravity field modeling — classic methods

Spherical harmonics

$$N(r,\theta,\lambda) = \frac{GM}{R\gamma}\sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^{n+1}\sum_{m=0}^{n}\bar{P}_{nm}(\cos\theta)\left[\Delta\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right]$$

Stokes integration in the frequency domain by 1D FFT

$$N(\varphi_P) = \frac{R\Delta\varphi\Delta\lambda}{4\pi\gamma_0}\mathcal{F}_1^{-1}\left[\sum_{\varphi_q=\varphi_1}^{\varphi_2}\mathcal{F}_1\{S(\Delta\lambda_{Pq})\}\cdot\mathcal{F}_1\{\Delta\bar{g}_q\cos\varphi_q\}\right]$$

Least-squares collocation (LSC)

$$\hat{N}_P = \mathbf{C}_{Pi}^{N\Delta g}\mathbf{C}_{ij}^{-1}\Delta g_i$$

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Gravity field modeling — radial basis functions (RBF)

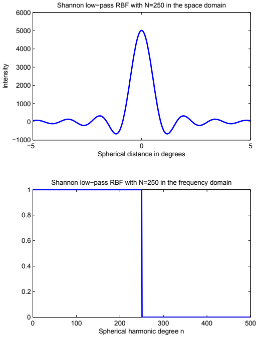
- RBFs are based on spherical harmonics (spectral representations):

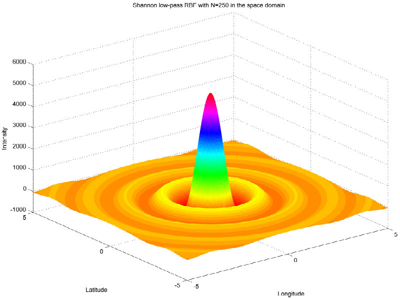
$$B(\vec{x}, \vec{x}_P) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi R^2} \left(\frac{R}{r}\right)^{n+1} B_n P_n(\vec{r}^T \vec{r}_P)$$

- The Legendre coefficients B_n define the kernel and reflect its behavior in the frequency domain
- The distance between the origin of the RBF on the sphere $\vec{x}_P$ and the computation point $\vec{x}$ is the only variable in the kernel
- The kernel reaches its maximum at $\vec{x}_P = \vec{x}$

Gravity field modeling — Shannon RBF

- $$B_n = 1 \quad \forall \quad n \in [0\,,\, N]$$





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Gravity field modeling — radial basis functions (RBF)

- RBFs are versatile → challenging!
- RBFs may be directly related to spherical harmonics:

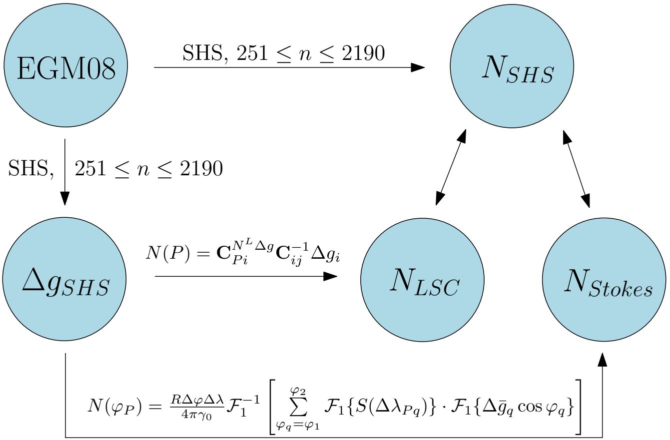
$$T(\vec{x}) = \frac{GM}{R} \sum_{k=0}^{\infty} d_k \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\frac{R}{r}\right)^{n+1} B_n P_n(\vec{r}^T \vec{r}_k)$$

- The RBF coefficients d_k constitute the RBF part which represents the signal and thus play a similar role as the spherical harmonic coefficients

Collocation versus Stokes integration

- Equivalent methods in the global case
- Collocation performs least-squares prediction over the entire Earth, and applies Stokes's formula to the globally continuous Δg function
- In a regional application, with near and distant zones, Stokes's formula is normally applied to a residual gravity signal in the near zone only
- In that case the models are not equivalent any more, and the cross-covariance function needs to be modified

Regional comparison of Stokes integration and collocation



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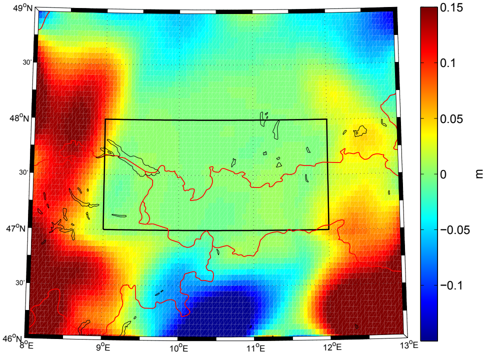
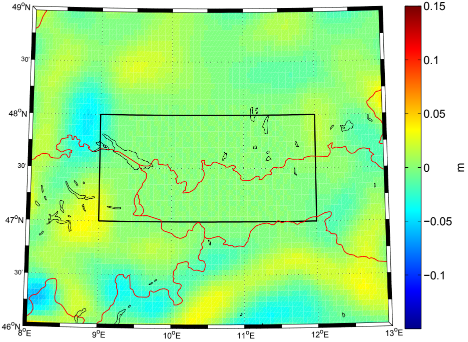
Regional comparison of Stokes integration and collocation — Alpine region



		max.	min.	mean	RMS
Data area	N_{Stokes}	0.405	-0.353	-0.013	0.140
	$N_{\text{SHS}} - N_{\text{Stokes}}$	0.446	-0.371	0.023	0.085
Target area	$N_{\text{SHS}} - N_{\text{Stokes}}$	0.010	-0.090	-0.003	0.010

Regional comparison of Stokes integration and collocation — Alpine region



		max.	min.	mean	RMS
	N_{LSC}	0.410	-0.355	-0.016	0.141
Data area	$N_{\text{SHS}} - N_{\text{LSC}}$	0.054	-0.059	0.000	0.014
Target area	$N_{\text{SHS}} - N_{\text{LSC}}$	0.032	-0.035	0.000	0.006

Inverse problems and regularization

- Why the transition to inverse problems and regularization?
- Inverse problems:
  - Determine spherical harmonic coefficients $\Delta\bar{C}_{nm}, \bar{S}_{nm}$ from observations (SHA)
  - Determine d-coefficients from observations (RBFA)
- The above problems are typically ill-conditioned, due to various reasons
- Both SHA and RBFA may be formulated as linear inverse problems which may be solved using parameter estimation methods

Inverse and ill-conditioned problems — Tikhonov regularization

- The system is solved as follows:

$$\hat{\vec{x}} = \left(\mathbf{A}^T \mathbf{P} \mathbf{A} + \mathbf{P}_{x_0}\right)^{-1} \left(\mathbf{A}^T \mathbf{P} \vec{l} + \mathbf{P}_{x_0} \vec{x}_0\right)$$

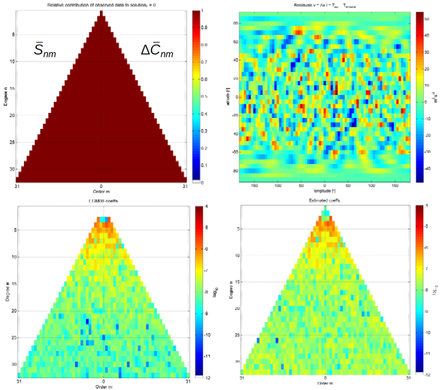
- SHA: the coefficients have expectation zero and are assumed to vary according to the signal degree variances computed from "true" EGM2008 coefficients
- RBFA: the coefficients have expectation zero and are assumed to vary according to a RMS value computed from "true" *d*-coefficients of EGM2008
- i.e., $\vec{x}_0 = 0$ and $\mathbf{P}_{x_0} = \alpha \mathbf{K}^{-1}$
- SHA: **K** is a diagonal matrix containing the degree variances
- RBFA: **K** is a diagonal matrix containing the RMS values

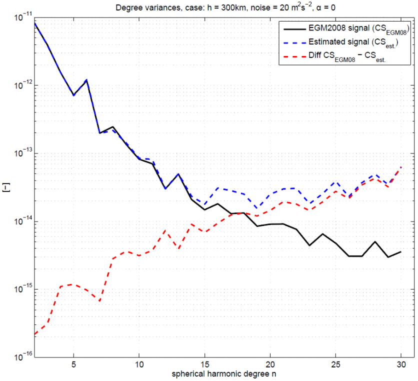
Spherical harmonic analysis

- Synthetic observations T, computed by SHS ($2 \leq n \leq 31$) on a regular latitude-longitude grid with 5° spacing using the global EGM2008 model
- In turn, the estimated SH coefficients were used to determine T, allowing the computation of "true" differences between the gravity fields
- Signal degree variances were computed from "true" EGM2008 coefficients as well as the estimated coefficients. Error degree variances were computed from their difference

Spherical harmonic analysis

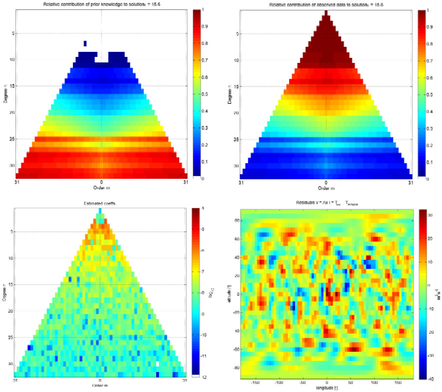
Noise $\sigma = 20\,\mathrm{m}^2\mathrm{s}^{-2}$, $h = 300\,\mathrm{km}$, no regularization $\alpha = 0$,
$\mathrm{cond}(\mathbf{N} + \alpha\mathbf{K}^{-1}) = 206$

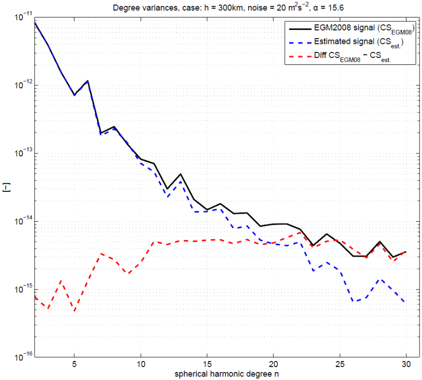




Spherical harmonic analysis

Noise $\sigma = 20\,\mathrm{m^2 s^{-2}}$, $h = 300\,\mathrm{km}$, regularization $\alpha_{L-curve} = 15.6$, $\mathrm{cond}(\mathbf{N} + \alpha \mathbf{K}^{-1}) = 37$





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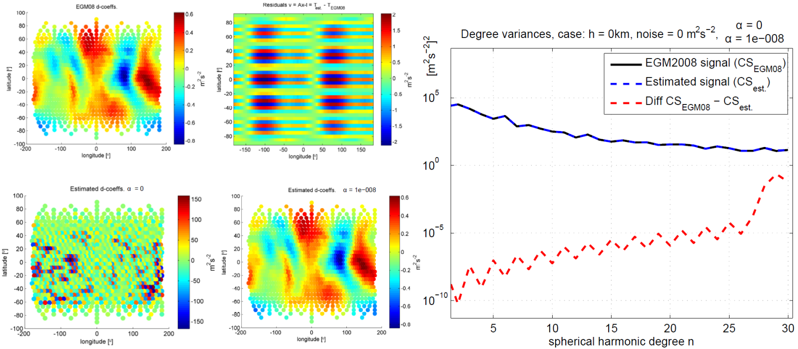
RBF analysis

- The RBFA case was set up similar to the SHA case
- “True” SH coefficients from EGM2008 were transferred to *d*-coefficients for the computation of gravity field differences
- Estimated *d*-coefficients were transferred to SH coefficients for signal and error degree variances plots
- Shannon RBFs were placed in a Reuter grid to avoid over-parametrization

RBF-analysis — difference between "mathematically correct" and "physically meaningful" coefficients

Noise-free observations, $h = 0$, $\alpha = 0 \rightarrow \mathrm{cond}(\mathbf{N} + \alpha\mathbf{K}^{-1}) = 6 \cdot 10^{18}$,
$\alpha_{small} \rightarrow \mathrm{cond}(\mathbf{N} + \alpha\mathbf{K}^{-1}) = 7 \cdot 10^{14}$



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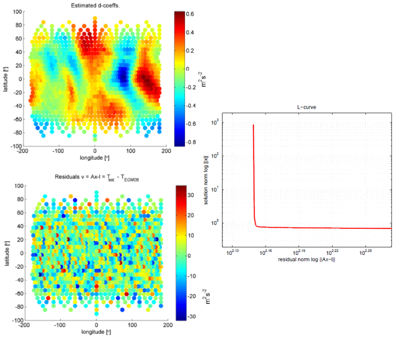
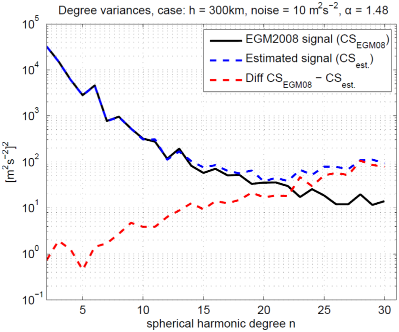
RBF-analysis, regularization using the L-curve method

Noise $\sigma = 10\,\mathrm{m^2 s^{-2}}$, $h = 300\,\mathrm{km}$, regularization
$\alpha_{L-curve} = 1.48 \rightarrow \mathrm{cond}(\mathbf{N} + \alpha \mathbf{K}^{-1}) = 599$
$(\alpha_{small} \rightarrow \mathrm{cond}(\mathbf{N} + \alpha \mathbf{K}^{-1}) = 9 \cdot 10^{10})$





Summary

- RBFA: ill-conditioned even in the ideal noise-free case
- We need “physically meaningful” d-coefficients
- Regularization with prior information on the unknowns seems to work, SHA and RBFA show similar traits
- Outlook: Apply this regularization scheme for regional gravity field modeling. Combine different observations. Model other quantities than T. Other RBFs more suitable for regional modeling?

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Thank you for your attention!